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Vector Area Theorem mapping in crystals and polarization stability of SIT-solitons

V. N. Lisin[1)]

Kazan Physical-Technical Institute RAS, 420029 Kazan, Russia

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The stability of polarization, areas, and number of self-induced transparency (SIT)-solitons on an output from the $LaF_3 : Pr^{3+}$ crystal is theoretically researched versus of the polarization direction and the area of the input linearly polarized laser pulse. For this purpose the Vector Area Theorem is rederived and two-dimensional Vector Area Theorem map is obtained. The map is governed by the crystal symmetry and take into account directions of the dipole matrix element vectors of the different site subgroups of optically excited ions. The Vector Area Theorem mapping of the time evolution of the laser pulse allows to highlight soliton polarization properties.

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For an isotropic medium stability properties of SIT-solitons are determined by the Area Theorem. The Area Theorem is the name given to a theoretical result that governs the coherent nonlinear transmission of short light pulses through isotropic materials that have an absorption resonance very near the frequency of the frequency of incident light, effectively two-level media. In 1967 McCall and Hahn [1] identified a new parameter (called "Area" and denoted by θ) of optical pulses travelling in such media, and then predicted that obeys the simple equation

$$\frac{\partial\theta}{\partial z} = -\frac{\alpha}{2}\sin\theta. \tag{1}$$

where α is the attenuation coefficient for the material. The two most striking consequences of the Area Theorem are: (i) pulses with special values of Area, namely all integer multiples of π, are predicted to maintain the same Area during propagation, and (ii) pulses with other values of Area must change in propagation until their Area reaches one of the special values. This property can be shown to be unstable for the odd multiples, but the even multiples enjoy the full immunity of the theorem. The Area Theorem (1) was derived for an isotropic material in which the dipole matrix element vector of any ion is parallel to the electrical field vector of the light pulse. By contrast, the direction of the dipole matrix element vector of any Pr^{3+} ion in LaF_3 is not dependent from the electrical field vector of the light pulse. So it is necessary to rederive Area Theorem taking into account directions of the dipole matrix element vectors of the different subgroups of Pr^{3+} ions. The Pr^{3+} ions in a LaF_3 unit cell can replace La^{3+} in six different types of sites $(\pm\alpha, \pm\beta, \pm\gamma)$. The local environment of any of them has C_2-symmetry. The six local C_2-symmetry axes are located in the plane normal to the C_3 axis and make the angle of $2\pi/6$ in this plane (Fig. 1). The electrical dipole matrix element vector of the Pr^{3+} ion (optical transition $\Gamma_1 \to \Gamma_1$)

$$\mathbf{p}_j = p\mathbf{e}_j, \cdots j = 1, \cdots, 6 \tag{2}$$

is directed [2] along of the local C_2-symmetry axis

$$\mathbf{e}_j = (\cos{(j\frac{2\pi}{6})}, \sin{(j\frac{2\pi}{6})}), \quad \mathbf{e}_z\cdot\mathbf{e}_j = 0. \tag{3}$$

where $\mathbf{e}_j$ is unit vector along the C_{2j} axis. Here axis Z is directed along the C_3-axis and axis X along α-axis. We define the Vector Area of light pulse as

$$\mathbf{\Theta} = \frac{p}{\hbar}\int_{-\infty}^{+\infty} dt\mathbf{E}(z,t), \tag{4}$$

where $\mathbf{E}(z,t)$ is vector amplitude of a light pulse, p is electrical dipole matrix element, $\hbar$ is the Plank's constant. Taking into account (2) and (3) and using arguments as Lamb [3] we can write the Vector Area Theorem as follows:

$$\frac{\partial\mathbf{\Theta}}{\partial z} = -\alpha\frac{1}{6}\sum_{j=1}^{6}\mathbf{e}_j\sin(\mathbf{\Theta}\cdot\mathbf{e}_j), \tag{5}$$

if a light pulse propagation is along C_3-axis. Here α is the linear attenuation coefficient for $LaF_3 : Pr^{3+}$. It can be seen from (5), as $\mathbf{\Theta}\to 0$, that

$$\frac{\partial\mathbf{\Theta}}{\partial z} = -\frac{\alpha}{2}\mathbf{\Theta}, \tag{6}$$

[1)] e-mail: vlisin@kfti.knc.ru

as it must for a small pulse Area.

Equating the right part of the equation (5) to zero we can find special values of the Vector Area where $\partial\mathbf{\Theta}/\partial z = 0$. But it to make much more obviously and more easy from the graphical representation. We can rewrite eq (5) as

$$\frac{\partial\mathbf{\Theta}}{\partial z} = \frac{\partial}{\partial\mathbf{\Theta}}\frac{\alpha}{6}\sum_{j=1}^{6}\cos(\mathbf{\Theta}\cdot\mathbf{e}_j), \tag{7}$$

And the problem is reduced to a determination of points in a two-dimensional plane, in which the function $\sum\cos(\mathbf{\Theta}\cdot\mathbf{e}_j)$ has extremes. The circles and triangles in Fig. 2 give the contour plot of this function. We easy find three types of special values of the Vector Area, namely

$$\mathbf{\Theta}_c = m\mathbf{\Theta}_+ + n\mathbf{\Theta}_-, \tag{8}$$

$$\mathbf{\Theta}_u = \mathbf{\Theta}_c + \mathbf{u}_j, \tag{9}$$

$$\mathbf{\Theta}_s = \mathbf{\Theta}_c + \mathbf{s}_j, \tag{10}$$

are predicted to maintain the same Vector Area during propagation. Here m and n are arbitrary integers and

$$\mathbf{\Theta}_+ = \frac{2\pi}{\cos(\pi/6)}\mathbf{k}_1, \qquad \mathbf{\Theta}_- = \frac{2\pi}{\cos(\pi/6)}\mathbf{k}_6; \tag{11}$$

$$\mathbf{s}_j = \frac{\pi}{\cos(\pi/6)}\mathbf{k}_j, \tag{12}$$

$$\mathbf{u}_j = \frac{\pi}{\cos^2(\pi/6)}\mathbf{e}_j, \tag{13}$$

where the unit vectors $\mathbf{e}_j$ (3) and $\mathbf{k}_j$ are directed along and between the C_2-axes accordingly:

$$\mathbf{k}_j = (\cos(j\frac{2\pi}{6} - \frac{\pi}{6}), \sin(j\frac{2\pi}{6} - \frac{\pi}{6})) \tag{14}$$

These peculiar points (8,9,10) give rise to a two-dimensional lattice in a $\mathbf{\Theta}$-phase plane with basis vectors $\mathbf{\Theta}_+$ and $\mathbf{\Theta}_-$ (11) as you can see in Fig. 2. The unit cell of the lattice is determined by symmetry of the crystal. It is a regular hexagon. The hexagon's centers are $\mathbf{\Theta}_c$ (8) (centers of the circles in Fig. 2). As measured from the hexagon center, coordinates of the six tops of the hexagon are $\mathbf{u}_j$ (13) (centers of the triangles in Fig. 2), coordinates of middles of the sides of a hexagon are $\mathbf{s}_j$ (12). As is easy to see from Eqs. (5,7) and definitions (8,9,10), that in a neighborhood of these singular points the Vector Area behaves as

$$\frac{\partial\mathbf{\Theta}}{\partial z} = -\frac{\alpha}{2}(\mathbf{\Theta} - \mathbf{\Theta}_c), \tag{15}$$

$$\frac{\partial\mathbf{\Theta}}{\partial z} = \frac{\alpha}{4}(\mathbf{\Theta} - \mathbf{\Theta}_u), \tag{16}$$

$$\frac{\partial\mathbf{\Theta}}{\partial z} = -\frac{\alpha}{6}(\mathbf{\Theta} - \mathbf{\Theta}_s), \tag{17}$$

if $\mathbf{\Theta} - \mathbf{\Theta}_s$ is directed along the side of a hexagon, and

$$\frac{\partial\mathbf{\Theta}}{\partial z} = +\frac{\alpha}{2}(\mathbf{\Theta} - \mathbf{\Theta}_s), \tag{18}$$

if $\mathbf{\Theta} - \mathbf{\Theta}_s$ is directed perpendicularly to the side of a hexagon. Therefore, for an absorbing (amplifying) medium with $\alpha > 0$ ($\alpha < 0$), the points (8) are type of a stable (unstable) knot, the points (9) are type of an unstable (stable) knot and the points (10) are type of a saddle in the $\mathbf{\Theta}$-phase plane. In further we shall explore a case of the absorbing medium with $\alpha > 0$. If an input pulse Vector Area falls inside an unit cell then the Vector Area must change in propagation until it reaches the unit cell center. If an input Vector Area is not equal to (9-10) and falls on a side of a hexagon then the Vector Area must change in a propagation until it reaches the middle of the gexagon side. It is necessary to mark that the Vector Area Theorem map (Fig. 2) allows easily to predict only the sum of the pulse vector areas on an output from the sample. To determine the number of the output SIT-solitons and their polarizations and areas we should solve the system of coupled Maxwell-Bloch equations.

The input pulse, which Vector Area is directed between the crystallographic axes and is equal, for example, to $\mathbf{\Theta}_+$, excite only four ($\pm\alpha, \pm\gamma$) ion subgroups. It is 2π-pulse for these ions. This pulse does not excite $\pm\beta$-ions, because $\mathbf{\Theta}_+ \perp \mathbf{e}_{\pm\beta}$. To the similarly previous case, the input pulse, which Vector Area is equal to $\mathbf{\Theta}_-$, is 2π-pulse for ($\pm\alpha, \pm\beta$)-ion subgroups and does not excite $\pm\gamma$-ions. If the input Vector Area is parallel to the $\mathbf{\Theta}_+$ ($\mathbf{\Theta}_-$) and fall inside the unit cell $\mathbf{\Theta}_c = m\mathbf{\Theta}_+$ ($\mathbf{\Theta}_c = m\mathbf{\Theta}_-$), then the time evolution of pulse, as it is easy to show, may be described by the inverse scattering method. The input pulse is splitting up on output, as for an isotropic medium, into m SIT- solitons, which Vector Area of any of them is $\mathbf{\Theta}_+$ ($\mathbf{\Theta}_-$). In further we shall refer to these solitons as $\mathbf{\Theta}_+$-solitons and $\mathbf{\Theta}_-$-solitons. If input Vector Area is not parallel to the $\mathbf{\Theta}_+$ ($\mathbf{\Theta}_-$) but fall inside the unit cell $\mathbf{\Theta}_c = m\mathbf{\Theta}_+$ ($\mathbf{\Theta}_c = m\mathbf{\Theta}_-$), then, as show numerical calculations, input pulse also is splitting up into m $\mathbf{\Theta}_+$ ($\mathbf{\Theta}_-$)-solitons on output.

If an input pulse Vector Area is directed along a crystallographic axis, for example the axis α, and is equal to

$$\mathbf{\Theta}_0 = \mathbf{\Theta}_+ + \mathbf{\Theta}_-, \tag{19}$$

then all $(\pm\alpha, \pm\beta, \pm\gamma)$ ions are excited. The input pulse is 2π-pulse for $(\pm\beta, \pm\gamma\ )$ ion subgroups and $4p$-pulse for $(\pm\alpha\ )$ ions. The time evolution of the pulse is not described by the inverse scattering method. The numerical calculations have shown that if an input Vector Area $\mathbf{\Theta}_{in}$ is parallel to $\mathbf{\Theta}_0$ and fall inside the unit cell $\mathbf{\Theta}_c = m\mathbf{\Theta}_0$, then the input pulse is splitting up into m SIT-solitons, which a Vector Area of any of them is $\mathbf{\Theta}_0$. In further we shall refer to these solitons as $\mathbf{\Theta}_0$-solitons. Let an input Vector Area is not parallel to $\mathbf{\Theta}_0$ and fall inside the unit cell $\mathbf{\Theta}_c = m\mathbf{\Theta}_0$. Then, as you can see in Figs. 2-3, a small change of an input pulse polarization reduces to that each of the $\mathbf{\Theta}_0$-solitons is splitting up into $\mathbf{\Theta}_+$-and $\mathbf{\Theta}_-$-solitons.Therefore a number of solitons and its polarization strongly depend on a direction of a vector $\mathbf{\Theta}_{in}$ concerning a crystallographic axis. This deduction also is valid and in a generally case when the input Vector Area full down inside the unit cell $\mathbf{\Theta}_c = m\mathbf{\Theta}_+ + n\mathbf{\Theta}_-$, where $m \neq n$. It is so because the unit cell center coordinates may be rewriting as $\mathbf{\Theta}_c = (m-n)\mathbf{\Theta}_+ + n\mathbf{\Theta}_0$ if $m > n$ or as $\mathbf{\Theta}_c = (n-m)\mathbf{\Theta}_- + m\mathbf{\Theta}_0$ if $n > m$. At first there are $(m-n)\mathbf{\Theta}_+$-solitons for $m > n$, or $(n-m)\mathbf{\Theta}_-$-solitons if $n > m$ on an output. After that the number of solitons appearing on an output depends on a direction of the vector $\mathbf{\Theta}_{in} - (m-n)\mathbf{\Theta}_+$ or $\mathbf{\Theta}_{in} - (n-m)\mathbf{\Theta}_-$ concerning a crystallographic axis. In a stable case the output solitons are $\mathbf{\Theta}_+$-solitons and $\mathbf{\Theta}_-$-solitons and its number is $(m+n)$.

For an amplifying medium $\mathbf{\Theta}_+$-solitons and $\mathbf{\Theta}_-$-solitons are unstable so the polarization of the output solitons must be directed along crystallographic axis in a stable case.

It is necessary to mark that for circular polarization of laser pulse the Area Theorem is (1) as in a case of isotropic medium.

To summarize, we have shown on an example of the model system $LaF_3 : Pr^{3+}$, that the Vector Area mapping of the pulse time evolution during a propagation is effective method to analyze the polarization properties of solitons.

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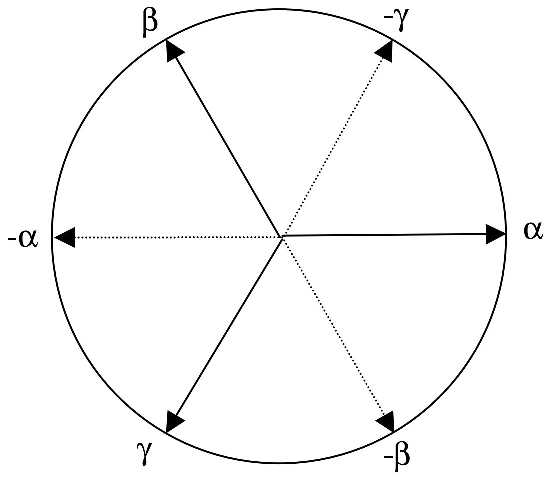
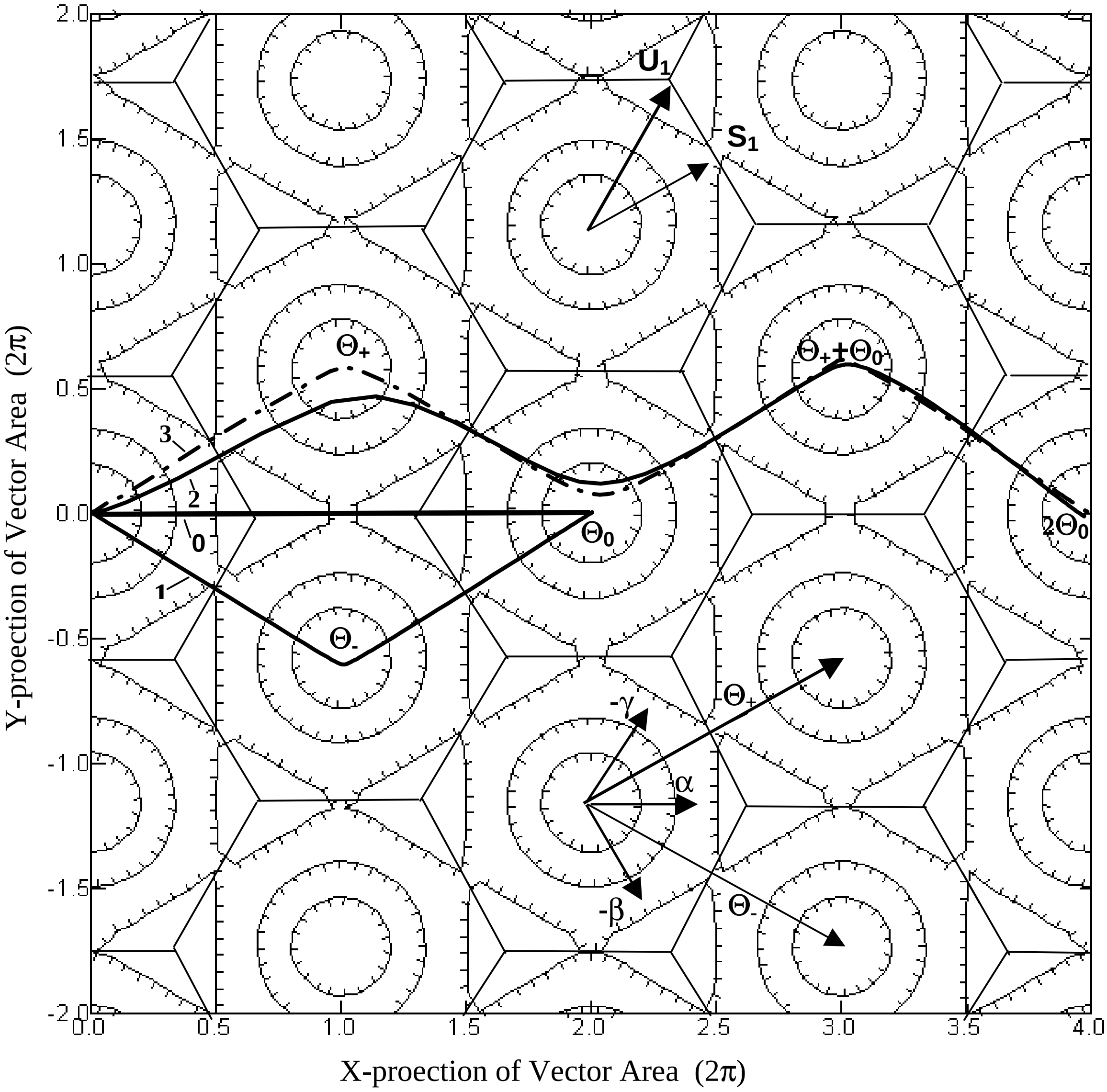
Fig. 1. The directions of the local C_2-symmetry axes for the different Pr^{3+} ion sites in the plane normal to the C_3- axis of the $LaF_3 : Pr^{3+}$ crystal.

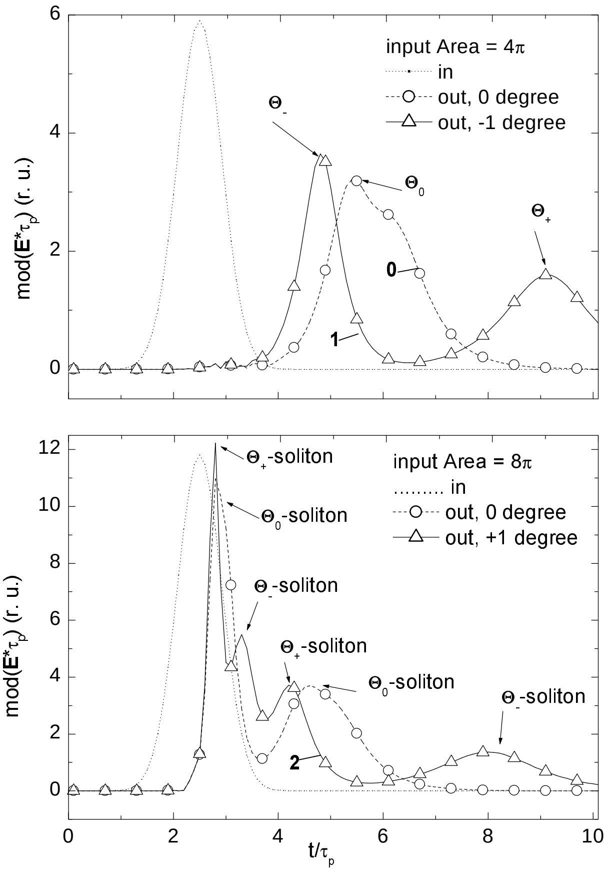
Fig. 2. Vector Area Theorem map. The $\mathbf{\Theta}/2\pi$ projections to axes X and Y are plotted on axes X and accordingly Y. The vector $\mathbf{s}_1$ (12) and $\mathbf{u}_1$ (13) are shown in an upper of the figure. There are the basis vectors $\mathbf{\Theta}_+$ and $\mathbf{\Theta}_-$ (11) and the unit vectors along of the local C_2-symmetry axes $(+\alpha, -\beta, -\gamma)$ in a down. Vector coordinates of some unit cell centers are also shown. Bold lines **0** and **1** are mappings of the time evolution of laser pulses which the input Vector Area is $mod(\mathbf{\Theta}_0) = 4\pi$ and the angles between the directions of the Vector Area and the crystallographic axis α are 0 and -1 degrees accordingly. In this case $\alpha L = 20$, where L is the sample length and α is the attenuation coefficient. The bold line **2** is the mapping of the time evolution of the laser pulse which the input Vector Area is $mod(2\mathbf{\Theta}_0) = 8\pi$ and the angle between the directions of Vector Area and the crystallographic axis α is $+1$ degree and $\alpha L = 40$. The bold line **3** is as **2** but $\alpha L = 80$.

Fig. 3. The time evolution of the amplitude of the laser pulses on the output of the sample. Values of the input Area, the angles between the directions of the Vector Area and the crystallographic axes α and the parameter αL for the curves **0**,**1** and **2** are the same as for the curves **0**,**1** and **2** in Fig. 2. The parameter value αL for the 0 degree curve in down of the figure is as for curve **2**. The dot line is input pulse, τ_p is input pulse duration. The Vector Area Theorem mapping (curves **1** and **2** in Fig. 2) allows easily to spot the polarizations and the areas of the solitons in curves **1** and **2** in this figure.


β
-γ
-α
α
γ
-β



U1
S1
Θ+
Θ++Θ0
Θ0
2Θ0
Θ-
-γ
α
-β
3
2
0
1
Y-proection of Vector Area (2π)
X-proection of Vector Area (2π)



input Area = 4π
in
out, 0 degree
out, -1 degree
Θ-
Θ0
Θ+
0
1
mod(E*τp) (r. u.)
input Area = 8π
.... in
out, 0 degree
out, +1 degree
Θ+-soliton
Θ0-soliton
Θ--soliton
Θ+-soliton
Θ0-soliton
Θ--soliton
2
mod(E*τp) (r. u.)
t/τp